3d Gauge Theories, Symplectic Duality and Knot Homology I

Tudor Dimofte
Notes by Qiaochu Yuan

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We want to study some 3d $N = 4$ supersymmetric QFTs. They won't be topological, but the supersymmetry will still help. We'll look in particular at boundary conditions and compactifications to 2d. Most of this material is relatively old by physics standards (known since the '90s). A motivation for discussing these theories here is that they seem closely related to

1. geometric representation theory,
2. geometric constructions of knot homologies, and
3. symplectic duality.

This story gives rise to some interesting mathematical predictions. On Tuesday we'll talk about what symplectic n-ality should mean. On Wednesday we'll give a new construction of projectives in variants of category O. On Thursday we'll talk about category O and knot homologies, and a new approach to both via 2d Landau-Ginzburg models.

1 Geometric representation theory

For starters, let $G = \mathrm{SL}_2$. The universal enveloping algebra of its Lie algebra is

$$U(\mathfrak{sl}_2) = \langle E, F, H \mid [E, F] = H, [H, E] = 2E, [H, F] = -2F\rangle. \tag{1}$$

The center is generated by the Casimir operator

$$\chi = EF + FE + \frac{1}{2}H^2. \tag{2}$$

The full category of $U(\mathfrak{sl}_2)$-modules is hard to work with. Bernstein, Gelfand, and Gelfand proposed that we work in a nice subcategory O of highest weight representations (on which E acts locally finitely or locally nilpotently, depending on the definition) decomposing as a sum

$$M = \bigoplus_\mu M_\mu \tag{3}$$

of generalized eigenspaces with respect to the action of the generator of the Cartan subalgebra H. Category O naturally decomposes into a direct sum $\bigoplus_\lambda O_\lambda$ of so-called blocks, where O_λ consists of representations on which χ acts by $\frac{\lambda^2-1}{2}$.

For generic λ (not an integer), O_λ contains two irreducible Verma modules $\Delta_{\lambda-1}, \Delta_{-\lambda-1}$ with no nontrivial Homs. Here Δ_r is the Verma module of highest weight r, with weights $r, r-2, \ldots$.

For $\lambda \in \mathbb{Z}$ (in general this will be replaced by a weight lattice), if $\lambda \geq 1$ then O_λ is a so-called regular block of category O. It is generated by the Verma (standard) modules $\Delta_{\lambda-1}, \Delta_{-\lambda-1}$, but the latter is now a submodule of the former; in particular the Hom space between them is $\mathbb{C}$, and $\Delta_{\lambda-1}$ is no longer irreducible (simple). There is a finite-dimensional irreducible module

$$L_{\lambda-1} = \Delta_{\lambda-1}/\Delta_{-\lambda-1} \tag{4}$$

of dimension λ, and $\Delta_{-\lambda-1}$ is irreducible.

There are also projective objects. Their defining property is that if P_μ is projective with weight μ, then taking (derived) $\mathrm{Hom}(P_\mu, M)$ projects onto the weight space M_μ. One of them is the Verma module $P_{\lambda-1} = \Delta_{\lambda-1}$. The other is an extension of one Verma module by the other one: $P_{-\lambda-1}$ fits into a short exact sequence

$$0 \to \Delta_{\lambda-1} \to P_{-\lambda-1} \to \Delta_{-\lambda-1} \to 0. \tag{5}$$

The Cartan H does not act diagonally, but has Jordan blocks.

O_λ secretly depends on two parameters. It depends on λ (really λ^2, the eigenvalue of the Casimir), but it secretly also depends on choosing to look at highest vs. lowest weights.

Recall that $U(\mathfrak{sl}_2)$ is naturally filtered; we will choose this filtration so that E, F, H have weight 2. We can also modify the universal enveloping algebra by taking $[E, F] = \hbar H$ for some additional parameter $\hbar$ which we will also take to have weight 2; this algebra $U(\mathfrak{sl}_2)_\hbar$ is graded.

1.1 Some geometry

Beilinson-Bernstein showed that O_λ is equivariant twisted D-modules on the flag variety $G/B \cong \mathbb{P}^1$. The twisting is given by λ thought of as an equivariant line bundle on G/B, and twisted D-modules means modules over the sheaf of differential operators acting on this line bundle. The equivariance will be explained later.

Braden-Licata-Proudfoot-Webster put this in a more general context. They consider conical symplectic resolutions

$$\pi : M \to M_0 \tag{6}$$

where M_0 is an affine cone and M is a smooth resolution of M_0; moreover, M_0 is algebraic (holomorphic) symplectic and more generally hyperkähler. M_0 has a $\mathbb{C}^*_\epsilon$-action such that the weight of the symplectic form Ω is 2, and the smooth resolution M has a compatible $U(1)_\epsilon$-action. Everything is done equivariantly with respect to this action.

(ϵ is a label here, used to distinguish this action from a different action. There is another $U(1)_\xi$-action which is a hyperkähler isometry, and it can be extended to a $\mathbb{C}^*_\xi$-action which is not a hyperkähler isometry but which is an holomorphic symplectic isometry.)

For example, we can take M to be the cotangent bundle $T^*(G/B)$ of a flag variety, M_0 the nilpotent cone N, and the map

$$\pi : T^*(G/B) \to N \tag{7}$$

to be the Springer resolution. In general, a choice of resolution roughly corresponds to a choice of $\lambda \in H^2(M, \mathbb{R})$ corresponding to the Kähler class $[\omega]$.

BLPW construct a sheaf D_λ giving a $\mathbb{C}^*_\epsilon$-equivariant quantization of (M, Ω). Its ring of global sections A_λ, which is a deformation of $\mathbb{C}[M]$ (which turns out to be $\mathbb{C}[M_0]$), has the property that

$$D_\lambda\text{-Mod} \cong A_\lambda\text{-Mod} \tag{8}$$

with some conditions. The D_λ-modules are restricted to have support on $M_+ \subset M$, which are points with limits under the $\mathbb{C}^*_\xi$-action, and the A_λ-modules are restricted so that the action of a subalgebra A_+ (the elements with positive weight under $U(1)_\xi$) is locally finite. This is the equivalent of the highest weight condition from before.

In our SL_2 example, we take $M = T^*(G/B) = T^*(\mathbb{P}^1)$. The $U(1)_\xi$-action comes from a $U(1)$-action on $\mathbb{P}^1$ and distinguishes a north and south pole. It also acts by rotation on the cotangent spaces at the north and south pole. The noncompact part of the $\mathbb{C}^*_\xi$-action pushes everything up to the north pole. M_+ consists of $\mathbb{CP}^1$ and the fiber at the south pole. $L_{\lambda-1}$ is supported on the base $\mathbb{P}^1$, whereas $\Delta_{\lambda-1}$ is supported on the base and the south pole fiber.

1.2 Dualities

Beilinson-Ginzburg-Soergel showed that there is a very nontrivial equivalence

$$O_G \cong O_{G^\vee} \tag{9}$$

where O_G is category O for G and $O_{G^\vee}$ is category O for the Langlands dual. When $G = \mathrm{SL}_2$ both live on $T^*(\mathbb{P}^1)$. One way to explain this is symplectic duality: there are pairs of conical symplectic resolutions $M, M^!$ which are Koszul dual. The Koszul duality

$$O_M \cong O_{M^!} \tag{10}$$

exchanges simples and projectives, and exchanges the role of the Casimir parameter λ and the highest weight parameter ξ (which determines what a highest weight is). (Here λ lives in $H^2(M, \mathbb{C})$ and ξ lives in the Cartan of the global hyperkähler isometry group of M. Part of the claim here is that these can be identified.) For example, $T^*(\mathbb{CP}^n)$ is related to a symplectic resolution of $\mathbb{C}^2/(\mathbb{Z}_{n+1})$ in this way. Physically these are two different branches of the moduli space of a 3d $N = 4$ theory.

2 Some physics

So much for math; let's do some physics. Classical field theory is about solutions to PDEs, e.g. the eigenvalue equation for the Laplacian on $\mathbb{R}^d$:

$$\partial_\mu \partial^\mu \varphi = m^2 \varphi. \tag{11}$$

Solutions to these equations are called classical fields. There may be several kinds of classical fields that can be coupled in the sense that we can insert interaction terms between them into the PDE.

The basic problem we want to solve is a scattering problem. After choosing a time coordinate, we want to pick an initial condition on $\mathbb{R}^{d-1}$ at some time and we'd like to solve the PDE to find out what happens at later times.

If $\mathbb{R}^d$ is secretly Minkowski space $\mathbb{R}^{d-1,1}$, we can ask that the PDE be invariant under the Poincaré group; for example, the eigenvalue equation for the Laplacian have this property.

We can require global (flavor) symmetries. For example, if $\varphi : \mathbb{R}^d \to \mathbb{C}$ is a complex function, we can change it by a phase

$$\varphi \mapsto e^{i\alpha}\varphi. \tag{12}$$

We can even require local (gauge) symmetries. Instead of taking the phase α to be fixed, we can consider

$$\varphi \mapsto e^{i\alpha(x)}\varphi. \tag{13}$$

Now to get a gauge invariant PDE we should consider the covariant derivative attached to a connection.

Finally, we can require that the field theory is Lagrangian. This means that the equations of motion are equivalent to finding the critical points of a functional, the Lagrangian L, given by integrating a Lagrangian density $\mathcal{L}$ over $\mathbb{R}^d$. All of these requirements have analogues in quantum theories.

Note that as $m \to 0$ in the equation

$$\partial_\mu \partial^\mu \varphi = m^2 \varphi \tag{14}$$

then the equation becomes a wave equation; this corresponds to a free field, which is massless. On the other hand, as $m \to \infty$ the equations of motion become $\varphi \equiv 0$; this is the infinitely massive theory, and corresponds to a vacuum.

Introduce an energy scale Λ. In units where $c = \hbar = 1$, the following units are the same:

$$\text{mass} = \text{energy} = \frac{1}{\text{distance}} = \frac{1}{\text{time}}. \tag{15}$$

In particular, m and Λ have the same units, so $\frac{m}{\Lambda}$ is a dimensionless version of mass and we can meaningfully talk about its size. At high energies (ultraviolet regime), the theory is effectively free, while at low energies (infrared regime), the theory is effectively stuck in a vacuum.